

A NOTE ON SOME APPLICATIONS OF THE EXTENSION OF ABEL'S LEMMA

ALEXANDER E PATKOWSKI

ABSTRACT. We offer some new applications of an extension of Abel's lemma, as well as its more general form established by Andrews and Freitas. A nice connection is established between this lemma and series involving the Riemann zeta function.

Keywords: Abel's lemma; Riemann zeta function; Series involving the zeta function.

2010 Mathematics Subject Classification 33D15;

1. INTRODUCTION

In a paper by Andrews and Freitas [4], the extension of Abel's lemma was further generalized and several new q -series were established. Recall that Abel's lemma is the simple result that $\lim_{z \rightarrow 1^-} (1-z) \sum_{n \geq 0} a_n z^n = \lim_{n \rightarrow \infty} a_n$. We use the shifted factorial notation $(a)_n = a(a+1) \cdots (a+n-1)$ in this paper [2]. Their result may be stated as follows.

Proposition 1.1. (*[4, Proposition 1.2]*) *Let $f(z) = \sum_{n \geq 0} \alpha_n z^n$ be analytic for $|z| < 1$, and assume that for some positive integer M and a fixed complex number α we have that (i) $\sum_{n \geq 0} (n+1)_M (\alpha_{M+n} - \alpha_{M+n-1})$ converges, and (ii) $\lim_{n \rightarrow \infty} (n+1)_M (\alpha_{M+n} - \alpha) = 0$. Then*

$$\frac{1}{M} \lim_{z \rightarrow 1^-} \left(\frac{\partial^M}{\partial z^M} (1-z) f(z) \right) = \sum_{n \geq 0} (n+1)_{M-1} (\alpha - \alpha_{n+M-1}).$$

The formula being generalized here is given in [3, Proposition 2.1], where it was used to find generating functions for special values for certain L -functions. A corollary of this extension of Abel's lemma was also given in [7].

In the work [1] we find a simple formula attributed there to a Christian Goldbach,

$$(1.1) \quad \sum_{n \geq 0} (1 - \zeta(k+2)) = -1.$$

Now it does not appear any connection has been made between the extension of Abel's lemma and this result, but as we shall demonstrate, it is a simple consequence of it. To this end, we shall prove some more general formulas in the next section which we believe are interesting applications of the Andrews-Freitas formula. For this, we will use a result from the work [6]. For some relevant series identities of a similar nature see also [5].

2. SOME NEW THEOREMS

This section establishes some interesting theorems, which we hope will add value to the Andrews-Freitas formula. For convenience in our proofs, we decided to write down a simple lemma.

Lemma 2.1. *If $f(z)$ has no factor $(1-z)^{-1}$, then we may write*

$$\lim_{z \rightarrow 1^-} \frac{\partial^M}{\partial z^M} (1-z)f(z) = -Mf^{M-1}(1).$$

Proof. Put $f_1(z) = (1-z)$, and $f_2(z) = f(z)$. Then by the Leibniz rule,

$$\begin{aligned} \lim_{z \rightarrow 1^-} \frac{\partial^M}{\partial z^M} (1-z)f(z) &= \lim_{z \rightarrow 1^-} \sum_{j \geq 0} \binom{M}{j} f_1^{(j)} f_2^{(M-j)} \\ &= \lim_{z \rightarrow 1^-} \binom{M}{1} f_1^{(1)} f_2^{(M-1)} \\ &= -M \lim_{z \rightarrow 1^-} f_2^{(M-1)}(z), \end{aligned}$$

because if $j = 0$ then the term in the sum, $f_1^{(0)}$, is 0 when $z \rightarrow 1^-$, and for $j > 1$, $f_1^{(j)} = 0$. \square

As usual, we denote γ to be Euler's constant [2]. We also define the polygamma function [2] to be the $(M+1)$ -th derivative of the logarithm of the Gamma function:

$$\psi^{(M)}(z) = \frac{\partial^{M+1}}{\partial z^{M+1}} (\log \Gamma(z)).$$

Theorem 2.2. *For positive integers M , we have that*

$$\sum_{n \geq 0} (n+1)_{M-1} (1 - \zeta(n+M+1)) = (-1)^M \sum_{j \geq 0} \binom{M-1}{j} j! \psi^{(M-j-1)}(1) + (M-1)! (-1)^M + \gamma (-1)^M (M-1)!.$$

Proof. First we write down the well-known Taylor expansion of the digamma function [1, 2], for $|z| < 1$,

$$(2.1) \quad \psi^{(0)}(z+1) = -\gamma - \sum_{k \geq 1} \zeta(k+1)(-z)^k.$$

It is a trivial exercise to re-write (2.1) as

$$(2.2) \quad -z^{-1}\psi^{(0)}(1-z) + z^{-1}\gamma = \sum_{k \geq 0} \zeta(k+2)z^k.$$

Inserting the functional equation for $\psi^{(0)}(z)$, given by [1, 2]

$$(2.3) \quad \psi^{(0)}(z+1) = \psi^{(0)}(z) + \frac{1}{z},$$

into (2.2) and multiplying by $(1-z)$ gives

$$(2.4) \quad -z^{-1}(1-z)(\psi^{(0)}(2-z) - (1-z)^{-1}) + z^{-1}(1-z)\gamma = (1-z) \sum_{k \geq 0} \zeta(k+2)z^k,$$

Now applying Proposition 1.1 with $\alpha_n = \zeta(n+2)$, and involving (2.4) gives the theorem after applying Lemma 2.1. \square

For $M = 1$ Theorem 2.2 specializes to Goldbach's formula (1.1).

Theorem 2.3. *For positive integers M and N , we have that*

$$\begin{aligned} \sum_{n \geq 0} (n+1)_{M-1} (n+M+1)^N (1 - \zeta(n+M+1)) &= \sum_{l \geq 1}^N S(N+1, l+1) (-1)^{l+1} g_{M,l} \\ &+ (-1)^M \sum_{j \geq 0} \binom{M-1}{j} j! \psi^{(M-j-1)}(1) + (M-1)! (-1)^M + \gamma (-1)^M (M-1)!, \end{aligned}$$

where for $l \geq 0$,

$$g_{M,l} := - \sum_{j \geq 0} \binom{M-1}{j} (-1)^{M-1-j} \psi^{(l+M-1-j)}(1) \frac{(l+1)!}{(l+1-j)!}.$$

Proof. From [6, Corollary 2], we find the delightful formula for integers $N \geq 1$ and $\Re(a) > 0$,

$$(2.5) \quad \sum_{k \geq 2} k^N z^k \zeta(k, a) = \sum_{l \geq 1}^N S(N+1, l+1) l! \zeta(l+1, a-z) z^{l+1} - z(\psi^{(0)}(a-z) - \psi^{(0)}(a)),$$

for $|z| < |a|$. Here $S(n, l)$ are the Stirling numbers of the second kind [2]. $\zeta(s, a)$ is the Hurwitz zeta function [2]. We have also corrected the stated formula by instead having $N \geq 1$. We have also shifted the sum by replacing l by $l+1$ for our convenience. Now $\lim_{n \rightarrow 0} \zeta(n, a) = 0$ if $a > 1$, 1 if $a = 1$, $+\infty$ if $0 < a < 1$. Hence

the formula (2.5) is of the type of interest to our study only if $a = 1$. So, in that case, we put $a = 1$, and re-write (2.5) as

$$(2.6) \quad \sum_{k \geq 2} k^N z^k \zeta(k) = \sum_{l \geq 1}^N S(N+1, l+1) l! \zeta(l+1, 1-z) z^{l+1} - z(\psi^{(0)}(1-z) - \psi^{(0)}(1)).$$

Differentiating (2.3) l times we get that

$$(2.7) \quad \psi^{(l)}(2-z) = \psi^{(l)}(1-z) + (1-z)^{-l-1}(-1)^l l!.$$

Now using equation [1, eq.(2.15)], we have

$$(2.8) \quad \sum_{k \geq 0} k^N z^k = \sum_{k \geq 1} k^N z^k = \sum_{l \geq 0}^N S(N+1, l+1) (-1)^l l! (1-z)^{-l-1} z^{l+1}.$$

Now $S(n, 1) = 1$ for all non-negative integers n , so we may write (2.8) for $N \geq 1$ as

$$(2.9) \quad \sum_{k \geq 0} k^N z^k = z(1-z)^{-1} + \sum_{l \geq 1}^N S(N+1, l+1) (-1)^l l! (1-z)^{-l-1} z^{l+1}.$$

Using $\psi^{(l)}(z) = (-1)^{l+1} l! \zeta(l+1, z)$, and (2.7), we re-write (2.6) as

$$(2.10) \quad \begin{aligned} \sum_{k \geq 2} k^N z^k \zeta(k) &= \sum_{l \geq 1}^N S(N+1, l+1) ((-1)^{l+1} \psi^{(l)}(2-z) + (1-z)^{-l-1} (-1)^l l! z^{l+1} \\ &\quad - z(\psi^{(0)}(1-z) - \psi^{(0)}(1))). \end{aligned}$$

Now comparing equation (2.9) with (2.10), and noting $\psi^{(0)}(1) = -\gamma$, we see that we have that

$$(2.11) \quad \begin{aligned} \sum_{k \geq 2} k^N z^k (\zeta(k) - 1) &= \sum_{l \geq 1}^N S(N+1, l+1) (-1)^{l+1} \psi^{(l)}(2-z) z^{l+1} + z - z(1-z)^{-1} \\ &\quad - z(\psi^{(0)}(1-z) + \gamma). \end{aligned}$$

Now we choose $\alpha_n = (n+2)^N (\zeta(n+2) - 1)$ and note that since 1 is removed from the first term in $\zeta(s)$ that $\lim_{n \rightarrow \infty} (n+2)^N (\zeta(n+2) - 1) = 0$, since exponential growth is faster than polynomial growth. The far right side of equation (2.11) may be construed as (2.2). Multiplying both sides by z^{-2} , and applying Proposition 1.1 we use the formula

$$(2.12) \quad \begin{aligned} \lim_{z \rightarrow 1^-} \frac{\partial^M}{\partial z^M} ((1-z) \psi^{(l)}(2-z) z^{l+1}) \\ = -M \lim_{z \rightarrow 1^-} \frac{\partial^{M-1}}{\partial z^{M-1}} (\psi^{(l)}(2-z) z^{l+1}) \end{aligned}$$

$$= -M \sum_{j \geq 0} \binom{M-1}{j} (-1)^{M-1-j} \psi^{(l+M-1-j)}(1) \frac{(l+1)!}{(l+1-j)!}.$$

We employed the trivial formula $\lim_{z \rightarrow 1^-} \frac{\partial^M}{\partial z^M} (z^l) = l!/(l-M)!$ in the last line. This proves the theorem after noting that the M -th derivative of $(1-z)z^{-1} - z^{-1} = -1$ is 0. \square

Note that since $N \geq 1$, Theorem 2.3 is not a generalization of Theorem 2.2 and so Theorem 2.2 is not redundant.

3. CONCLUSION

The conclusion we have come to here is that the summation formula that was established to prove interesting q -series identities may also be used to prove identities for series involving the Riemann zeta function. Some further interest should be directed toward finding expressions for sums of the form

$$\sum_{n \geq 0} a_n (L(n + \sigma + 1) - 1),$$

where the a_n are appropriately chosen for the series to converge, and $L(s)$ is a Dirichlet series which is assumed to have its first term to be 1 and converges when $\Re(s) > \sigma$. We believe this is a curious incidence where attractive results in one area of mathematics may be grouped as a consequence of a formula which has produced attractive results in another area.

REFERENCES

- [1] V. S. Adamchik, H. M. Srivastava, *Some series of the zeta and related functions*, Analysis (Munich) 18 (1998) 131–144.
- [2] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, New York, 1999.
- [3] G. E. Andrews, J. Jimenez-Urroz and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. Journal 108 (2001), 395–419.
- [4] G. E. Andrews, and P. Freitas, *Extension of Abel's lemma with q-series implications*, Ramanujan J. 10 (2005), 137–152.
- [5] J. Choi, Y. J. Cho, H. M. Srivastava, *Series involving the Zeta function and multiple Gamma functions*, Appl. Math. Comput., 159 (2004), no. 2, 509–537.
- [6] M. Hashimoto, S. Kanemitsu, Y. Tanigawa, M. Yoshimoto, W.-P. Zhang, *On some slowly convergent series involving the Hurwitz zeta-function*, J. Computational Applied Math. 160 (2003), 113–123.

- [7] A. Patkowski, *An Observation on the extension of Abel's Lemma*, *Integers* 10 (2010), 793–800.

1390 Bumps River Rd.

Centerville, MA 02632

USA

E-mail: alexpatk@hotmail.com